Computability of Haar Averages*

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Definition 1. *For any measure space $(X, \mathcal{B}(X), \mu)$ with a Borel probability measure, an average means the functional $C(X) \to \mathbb{R}$ s.t. maps f to $\int_X f d\mu$.*

It is known that the average over the real interval $[0, 1]$ is computable. For averages over more general spaces, it is known that Borel probability measure can be represented by a probabilistic process. And if we receive it as an input, we can compute the average over a topological space with respect to this measure [1]. However, Haar's theorem states that on any compact topological group, there exists a unique probability measure on the Borel subsets which is left-translation-invariant and regular. It implies that the measure with such conditions is represented implicitly by the space, and no need to receive it as an input. From now on, we call the average with the Haar probability measure as the Haar Average.

In this talk, I prove that Haar Averages are computable under natural and mild assumption. It generalizes average over the real interval $[0, 1]$ and differs from accepting the measure as an input. For compact metric spaces, [2] states that assuming computability of "separation bound" is natural. I followed this spirit and assumed that the sizes of maximal packings is computable where

Definition 2. *For any compact metric space (X, d) and its subset $T \subseteq X$,*

(1) T is called a n-packing if $x \in T, y \in T \Rightarrow x = y \vee d(x, y) > 2^{-n}$
(2) $\kappa_X : \mathbb{N} \to \mathbb{N}$ is called the sizes of maximal packings if

$$\kappa_X(n) = \max_{T \text{ is a } n\text{-packing}} |T|$$

(3) T is called a maximal n-packing if it is a n-packing and $|T| = \kappa_X(n)$.

These definitions are very similar to [2] and thus natural, but slightly different. I also assumed the space is a computable metric space with bi-invariant metric, which means there is an encoding of some countable dense set and metric function, which takes this encodings as an input and outputs the distance, is computable.

Theorem 3. *Assume that the space $(X, d, \circ)$ is a compact topological group and a computable metric space with bi-invariant metric. Assume that the sizes of maximal packings is computable. Then the Haar Average functional is computable.*

In the proof, measures of sets are approximated by the following lemma:

Lemma 4. *For measurable set $U \subseteq X, n \in \mathbb{N}$ and maximal n-packing T,*

$$\frac{\left|T \cap \overline{B}_{-2^{-n+1}}(U)\right|}{|T|} \leq \mu(U) \leq \frac{\left|T \cap \overline{B}_{2^{-n+1}}(U)\right|}{|T|}$$

where $\overline{B}_r(U) := \bigcup_{x \in U} \overline{B}_r(x)$ and $\overline{B}_{-r}(U) := \{x \in U : \overline{B}_r(x) \subseteq U\}$.

References

1. M. Schröder and A. Simpson, "Representing probability measures using probabilistic processes," *J. Complexity*, vol. 22, no. 6, pp. 768–782, 2006.
2. K. Weihrauch, "Computational complexity on computable metric spaces," *Math. Log. Q.*, vol. 49, no. 1, pp. 3–21, 2003.

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