Improving Approximations by Taylor Models in Computable Analysis *

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Abstract

This paper deals with the theoretical background for recent improvements on the iRRAM software [MU01] for exact real arithmetic (ERA) and extends [BKM16]. ERA can be viewed as an implemented version of the theory of computability on non-denumerable sets, often called Type-2-Theory of Effectivity (TTE, [Wei00,BHW08]). The techniques discussed in this paper can also easily be applied to other software for exact real computations like [BK08,DFKT14].

The set of real numbers is a non-denumerable set, thus the ordinary computability theory using Turing machines (or equivalent concepts) can not be applied directly. Instead, oracle Turing machines or Type-2 Turing machines have to be used, which additionally introduce the idea of infinite computations transforming converging sequences of approximations into new converging sequences.

The approximations considered in TTE usually come from naïve interval arithmetic. However, Taylor models (e.g. [MB01]) as the current state of art in interval arithmetic have rarely been considered in this area. There are only few papers like [Bla05,Spa10,KVO14,DFKT14] pointing in this direction.

Taylor models are polynomials approximating functions on a restricted domain with coefficients usually based on 64-bit double precision numbers according to the standard IEEE 754-2008. The main focus is to get good approximations even in this restricted finite case, where convergence cannot even be discussed.

Compared to plain intervals, algorithms on polynomials could require a higher running time. However, when the output is required to be accurate to \(n\) digits, the reduced loss of accuracy when using TMs leads to require less precise input. Therefore the basic operations on such multiple-precision numbers are faster. This effect can counteract the increased complexity on polynomials and in total lead to shorter running times to obtain the desired output accuracy. We have observed this effect when solving IVPs [BKM15].

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In [BKM16] we took a quite broad look on ‘wrapping’ representations for vectors of real numbers, where the use of Taylor models is just one of many possibilities for the definition of converging sequences of approximations. We were able to show that in general, all these approaches are equivalent under reasonable conditions.

Though constructive, this equivalence is not used in practice, where algorithms on real numbers are implemented using the limit of approximations computed by the same algorithm on intervals, Taylor models or on other notations of countable sets dense in $\mathbb{R}$ instead. Thus, in order to compare accuracies between these approaches in a practical setting the algorithm itself has to be regarded as fixed.

In this paper we concentrate on representations derived from linear Taylor models (in the form of generalized interval arithmetic [Han75]). One important goal is to show how algorithms for ERA using Taylor models can be derived from ordinary interval based versions. In addition we discuss in which cases these new algorithms are superior to the previous ones. A complexity theory in the sense of [Ko91] does not exist yet, however, and is the goal of our ongoing research.

References


